

Asymptotic analysis of the one-dimensional quantum walks by the Tsallis and Rényi entropies

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Abstract. The Tsallis and Rényi entropies are important quantities in the information theory, statistics and related fields because the Tsallis entropy is an one parameter generalization of the Shannon entropy and the Rényi entropy includes several useful entropy measures such as the Shannon entropy, Min-entropy and so on, as special choices of its parameter. On the other hand, the discrete-time quantum walk plays important roles in various applications, for example, quantum speed-up algorithm and universal computation. In this paper, we show limiting behaviors of the Tsallis and Rényi entropies for discrete-time quantum walks on the line which are starting from the origin and defined by arbitrary coin and initial state. The results show that the Tsallis entropy behaves in polynomial order of time with the parameter dependent exponent while the Rényi entropy tends to infinity in logarithmic order of time independent of the choice of the parameter. Moreover, we show the difference between the Rényi entropy and the logarithmic function characterizes by the Rényi entropy of the limit distribution of the quantum walk. In addition, we show an example of asymptotic behavior of the conditional Rényi entropies of the quantum walk.

1 Introduction

The Tsallis entropy [22] and the Rényi entropy [19] are important quantities in the information theory, statistics and related fields. The Tsallis entropy is viewed as an one parameter generalization of the Shannon entropy. Also the Rényi entropy includes several useful entropy measures such as the Shannon entropy, Min-entropy and so on, as special cases. In the information theory, information measures such as entropies play a significant role because they measure the quantity of information [11]. In addition, the (relative) Shannon entropy gives a meaning of the rate function of the Large Deviation Principle (LDP) for random walks.

On the other hand, the discrete-time quantum walks (DTQWs) have been attractive research topics in the last decade as quantum counterparts of the random walks [1, 4, 12, 13, 16, 18, 23]. As the random walk plays important roles in various fields, DTQW also plays such roles in various applications, for example, quantum speed-up algorithm [2, 3, 9, 20] and universal quantum computation [8, 17]. Recently, the LDP for the one-dimensional DTQW have been shown in [21]. In the present stage, the entropic meaning of the rate function for the LDP for DTQW is unknown. Therefore it is important to make clear the roles of entropies in DTQWs.

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In this paper, we investigate asymptotic behaviors of the Tsallis and Rényi entropies of DTQWs on the line as a fundamental study. The result for the Rényi entropy is a generalization of the result for the Shannon entropy given by [10]. Therefore it is consistent for the numerical results of [6, 7] for the Shannon entropy.

The rest of this paper is organized as follows. In Sect. 2, we give the definition of the DTQW and state our results for the Tsallis and Rényi entropies (Theorem 2.1) and the conditional Rényi entropies (Corollary 2.2). The proof of the Theorem 2.1 is presented in Sect. 3. In this paper, we use two for the base of logarithm functions but the choice of the base is not essential.

2 Definition and results

The discrete-time quantum walk is a quantum counterpart of the classical random walk which has additional degree of freedom called chirality. For DTQW on the line \mathbb{Z} where \mathbb{Z} is the set of integers, the chirality takes two values left and right, and it means the direction of the motion of the walker. Now we define the following two dimensional vectors:

$$|L\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |R\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where L and R refer to the left and right chirality state, respectively. At each time step, the walker moves one step to the left if it has the left chirality, and if it has the right chirality, it moves one step to the right.

Let $U(2)$ denote the set of 2×2 unitary matrices. The time evolution of the DTQW on \mathbb{Z} is defined by

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(2),$$

with $a, b, c, d \in \mathbb{C}$ where \mathbb{C} is the set of complex numbers. The unitarity of U gives

$$|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1, \quad a\bar{c} + b\bar{d} = 0, \quad c = -\Delta\bar{b}, \quad d = \Delta\bar{a},$$

where \bar{z} is the complex conjugate of $z \in \mathbb{C}$ and $\Delta = \det U = ad - bc$ with $|\Delta| = 1$. In order to define the dynamics of the model, we divide U into two matrices:

$$P = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix},$$

with $U = P + Q$. The matrix P (resp. Q) represents the weight of the walker's movement to the left (resp. right) at each time step. Let $\Xi_n(l, m)$ denote the sum of all paths starting from the origin in the trajectory consisting of l steps left and m steps right. In fact, for time $n = l + m$ and position $x = -l + m$, we have

$$\Xi_n(l, m) = \sum_{l_j, m_j} P^{l_1} Q^{m_1} P^{l_2} Q^{m_2} \dots P^{l_{n-1}} Q^{m_{n-1}} P^{l_n} Q^{m_n},$$

where the summation is taken over all integers $l_j, m_j \geq 0$ satisfying $l_1 + \dots + l_n = l$, $m_1 + \dots + m_n = m$, $l_j + m_j = 1$. We should note that the definition gives

$$\Xi_{n+1}(l, m) = P \Xi_n(l-1, m) + Q \Xi_n(l, m-1).$$

For example, in the case of $l = 3$, $m = 1$, we have

$$\Xi_4(3, 1) = QP^3 + PQP^2 + P^2QP + P^3Q.$$

The set of initial qubit states at the origin for the DTQW is given by

$$\Phi = \{\varphi = \alpha|L\rangle + \beta|R\rangle \in \mathbb{C}^2 : |\alpha|^2 + |\beta|^2 = 1\}. \quad (2.1)$$

The probability that a quantum walker is in position x at time n starting from the origin with $\varphi \in \Phi$ is defined by

$$P(X_n^\varphi = x) = \|\Xi_n(l, m)\varphi\|^2,$$

where $n = l + m$ and $x = -l + m$.

The Tsallis entropy $T_{\hat{\alpha}}(X)$ and the Rényi entropy $R_{\hat{\alpha}}(X)$ of order $\hat{\alpha} \in [0, \infty) \setminus \{1\}$ for a random variable X taking values in a finite set \mathcal{X} are given by

$$T_{\hat{\alpha}}(X) = \frac{1}{1 - \hat{\alpha}} \left(\sum_{x \in \mathcal{X}} P_X(x)^{\hat{\alpha}} - 1 \right),$$

$$R_{\hat{\alpha}}(X) = \frac{1}{1 - \hat{\alpha}} \log_2 \left\{ \sum_{x \in \mathcal{X}} P_X(x)^{\hat{\alpha}} \right\},$$

where P_X is a probability measure on \mathcal{X} . Let $P_n^\varphi(x) = P(X_n^\varphi = x)$. Here we define the Tsallis entropy of order $\hat{\alpha} \in [0, \infty) \setminus \{1\}$ for the DTQW at time n as follows:

$$T_{\hat{\alpha}}^\varphi(n) = T_{\hat{\alpha}}(X_n^\varphi) = \frac{1}{1 - \hat{\alpha}} \left(\sum_{x=-n}^n P_n^\varphi(x)^{\hat{\alpha}} - 1 \right). \quad (2.2)$$

Also we define the Rényi entropy of order $\hat{\alpha} \in [0, \infty) \setminus \{1\}$ for the DTQW at time n as follows:

$$R_{\hat{\alpha}}^\varphi(n) = R_{\hat{\alpha}}(X_n^\varphi) = \frac{1}{1 - \hat{\alpha}} \log_2 \left\{ \sum_{x=-n}^n P_n^\varphi(x)^{\hat{\alpha}} \right\}. \quad (2.3)$$

Note that the limit $\lim_{\hat{\alpha} \rightarrow 1} R_{\hat{\alpha}}^\varphi(n)$ is the Shannon entropy $S_n^\varphi = -\sum_{x=-n}^n P_n^\varphi(x) \log_2 P_n^\varphi(x)$. Also $R_\infty^\varphi := \lim_{\hat{\alpha} \rightarrow \infty} R_{\hat{\alpha}}^\varphi(n)$ is equal to $-\log \max_{-n \leq x \leq n} P_n^\varphi(x)$ the min-entropy. Remark that the Tsallis entropy and the Rényi entropy defined by Eqs. (2.2) and (2.3) have the following one-to-one correspondence:

$$T_{\hat{\alpha}}^\varphi(n) = \frac{1}{1 - \hat{\alpha}} \left(2^{(1-\hat{\alpha})R_{\hat{\alpha}}^\varphi(n)} - 1 \right).$$

In this paper, we show the following long-time behavior of the Tsallis entropy and the Rényi entropy:

THEOREM 2.1 *If the DTQW is determined by U with $abcd \neq 0$, then we have*

$$\lim_{n \rightarrow \infty} \left\{ \frac{T_{\hat{\alpha}}^\varphi(n) + 1/(1 - \hat{\alpha})}{(n/2)^{1-\hat{\alpha}}} - \frac{1}{1 - \hat{\alpha}} \right\} = \frac{1}{1 - \hat{\alpha}} \left(\int_{-|a|}^{|a|} f(x)^{\hat{\alpha}} dx - 1 \right),$$

and

$$\lim_{n \rightarrow \infty} \log_2(n/2) \left(\frac{R_{\hat{\alpha}}^\varphi(n)}{\log_2(n/2)} - 1 \right) = \frac{1}{1 - \hat{\alpha}} \log_2 \left\{ \int_{-|a|}^{|a|} f(x)^{\hat{\alpha}} dx \right\} =: R_{\hat{\alpha}}^\varphi(\infty), \quad (2.4)$$

for each $\hat{\alpha} \in [0, \infty) \setminus \{1\}$. Here

$$f(x) = \frac{|b|}{\pi(1-x^2)\sqrt{|a|^2-x^2}} \left\{ 1 - \left(|\alpha|^2 - |\beta|^2 + \frac{a\alpha\bar{b}\bar{\beta} + \bar{a}\bar{\alpha}b\beta}{|a|^2} \right) x \right\}.$$

Theorem 2.1 shows that the asymptotic behavior of the Tsallis entropy $T_{\hat{\alpha}}^\varphi(n)$ is governed by $(n/2)^{1-\hat{\alpha}}$ while the Rényi entropy $R_{\hat{\alpha}}^\varphi(n)$ tends to infinity in order $\log_2(n/2)$ independent of choice of the parameter $\hat{\alpha}$. Moreover, the difference between the Rényi entropy $R_{\hat{\alpha}}^\varphi(n)$ and $\log_2(n/2)$ is measured by the Rényi entropy with related to $f(x)$ the limit density function of X_n^φ/n which is obtained by [14, 15]. This situation is consistent with the Shannon entropy case [10]. It is observed that asymptotic behaviors of the Tsallis and Rényi entropies are strongly dependent on that of X_n^φ/n in any choice of the parameter $\hat{\alpha}$.

It is natural to consider the conditional entropy for DTQW for the next step. In this paper, we consider the conditional Rényi entropies. As it is already mentioned by [11], from the axiomatic point of view, there

are five types of definitions of the conditional Rényi entropies of order $\hat{\alpha} \in [0, \infty) \setminus \{1\}$ for the DTQW at time n as follows:

$$\begin{aligned}
R_{\hat{\alpha}}^C(X_n^\varphi|Y) &= \sum_{y \in \mathcal{Y}} P_Y(y) R_{\hat{\alpha}}(X_n^\varphi|Y=y), \\
R_{\hat{\alpha}}^{JA}(X_n^\varphi|Y) &= R_{\hat{\alpha}}(X_n^\varphi, Y) - R_{\hat{\alpha}}(Y), \\
R_{\hat{\alpha}}^{RW}(X_n^\varphi|Y) &= \frac{1}{1-\hat{\alpha}} \max_{y \in \mathcal{Y}} \left[\log_2 \left\{ \sum_{x=-n}^n P_{X_n^\varphi|Y}(x|y)^{\hat{\alpha}} \right\} \right], \\
R_{\hat{\alpha}}^A(X_n^\varphi|Y) &= \frac{\hat{\alpha}}{1-\hat{\alpha}} \log_2 \left[\sum_{y \in \mathcal{Y}} P_Y(y) \left\{ \sum_{x=-n}^n P_{X_n^\varphi|Y}(x|y)^{\hat{\alpha}} \right\}^{1/\hat{\alpha}} \right], \\
R_{\hat{\alpha}}^H(X_n^\varphi|Y) &= \frac{1}{1-\hat{\alpha}} \log_2 \left[\sum_{y \in \mathcal{Y}} P_Y(y) \left\{ \sum_{x=-n}^n P_{X_n^\varphi|Y}(x|y)^{\hat{\alpha}} \right\} \right],
\end{aligned}$$

where Y is a random variable taking values in a finite set \mathcal{Y} .

Now we examine the case that the initial qubit state φ of DTQW is randomly chosen by a random variable Y with some probability measure P_Y on $\mathcal{Y} \subset \Phi$ defined by Eq. (2.1). In this setting, the conditional probability measure $P_{X_n^\varphi|Y}$ is the same as P_n^φ on the condition $\{Y = \varphi\}$. In addition, $\log_2 \left\{ \sum_{x=-n}^n P_n^\varphi(x)^{\hat{\alpha}} \right\} = (1-\hat{\alpha})R_{\hat{\alpha}}^\varphi(n)$ by the definition of the Rényi entropy Eq. (2.3). Then these conditional entropies are calculated by the following forms:

$$\begin{aligned}
R_{\hat{\alpha}}^C(X_n^\varphi|Y) &= \sum_{\varphi \in \mathcal{Y}} P_Y(\varphi) R_{\hat{\alpha}}^\varphi(n), \\
R_{\hat{\alpha}}^{JA}(X_n^\varphi|Y) &= \frac{1}{1-\hat{\alpha}} \log_2 \left\{ \sum_{\varphi \in \mathcal{Y}} P_Y(\varphi)^{\hat{\alpha}} \cdot 2^{(1-\hat{\alpha})R_{\hat{\alpha}}^\varphi(n)} \right\} - R_{\hat{\alpha}}(Y), \\
R_{\hat{\alpha}}^{RW}(X_n^\varphi|Y) &= \begin{cases} \max_{\varphi \in \mathcal{Y}} \{R_{\hat{\alpha}}^\varphi(n)\}, & \text{if } 0 \leq \hat{\alpha} < 1, \\ \min_{\varphi \in \mathcal{Y}} \{R_{\hat{\alpha}}^\varphi(n)\}, & \text{if } 1 < \hat{\alpha}, \end{cases} \\
R_{\hat{\alpha}}^A(X_n^\varphi|Y) &= \frac{\hat{\alpha}}{1-\hat{\alpha}} \log_2 \left\{ \sum_{\varphi \in \mathcal{Y}} P_Y(\varphi) \cdot 2^{((1-\hat{\alpha})/\hat{\alpha})R_{\hat{\alpha}}^\varphi(n)} \right\}, \\
R_{\hat{\alpha}}^H(X_n^\varphi|Y) &= \frac{1}{1-\hat{\alpha}} \log_2 \left\{ \sum_{\varphi \in \mathcal{Y}} P_Y(\varphi) \cdot 2^{(1-\hat{\alpha})R_{\hat{\alpha}}^\varphi(n)} \right\}.
\end{aligned}$$

As a consequence, we have the following corollary by using Theorem 2.1:

COROLLARY 2.2 *If the DTQW is determined by U with $abcd \neq 0$, then we have the limits of the conditional*

Rényi entropies for each $\hat{\alpha} \in [0, \infty) \setminus \{1\}$.

$$\begin{aligned}
\lim_{n \rightarrow \infty} \log_2(n/2) \left(\frac{R_{\hat{\alpha}}^C(X_n^\varphi|Y)}{\log_2(n/2)} - 1 \right) &= \sum_{\varphi \in \mathcal{Y}} P_Y(\varphi) R_{\hat{\alpha}}^\varphi(\infty), \\
\lim_{n \rightarrow \infty} \log_2(n/2) \left(\frac{R_{\hat{\alpha}}^{JA}(X_n^\varphi|Y)}{\log_2(n/2)} - 1 \right) &= \frac{1}{1 - \hat{\alpha}} \log_2 \left\{ \sum_{\varphi \in \mathcal{Y}} P_Y(\varphi)^{\hat{\alpha}} \cdot 2^{(1-\hat{\alpha})R_{\hat{\alpha}}^\varphi(\infty)} \right\} - R_{\hat{\alpha}}(Y), \\
\lim_{n \rightarrow \infty} \log_2(n/2) \left(\frac{R_{\hat{\alpha}}^{RW}(X_n^\varphi|Y)}{\log_2(n/2)} - 1 \right) &= \begin{cases} \max_{\varphi \in \mathcal{Y}} \{R_{\hat{\alpha}}^\varphi(\infty)\}, & \text{if } 0 \leq \hat{\alpha} < 1, \\ \min_{\varphi \in \mathcal{Y}} \{R_{\hat{\alpha}}^\varphi(\infty)\}, & \text{if } 1 < \hat{\alpha}, \end{cases} \\
\lim_{n \rightarrow \infty} \log_2(n/2) \left(\frac{R_{\hat{\alpha}}^A(X_n^\varphi|Y)}{\log_2(n/2)} - 1 \right) &= \frac{\hat{\alpha}}{1 - \hat{\alpha}} \log_2 \left\{ \sum_{\varphi \in \mathcal{Y}} P_Y(\varphi) \cdot 2^{((1-\hat{\alpha})/\hat{\alpha})R_{\hat{\alpha}}^\varphi(\infty)} \right\}, \\
\lim_{n \rightarrow \infty} \log_2(n/2) \left(\frac{R_{\hat{\alpha}}^H(X_n^\varphi|Y)}{\log_2(n/2)} - 1 \right) &= \frac{1}{1 - \hat{\alpha}} \log_2 \left\{ \sum_{\varphi \in \mathcal{Y}} P_Y(\varphi) \cdot 2^{(1-\hat{\alpha})R_{\hat{\alpha}}^\varphi(\infty)} \right\},
\end{aligned}$$

where $R_{\hat{\alpha}}^\varphi(\infty)$ is defined by Eq. (2.4).

3 Proof of Theorem 2.1

In this section we assume $abcd \neq 0$. We consider the following four matrices:

$$P = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}, \quad R = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}.$$

Put $x \wedge y = \min\{x, y\}$. For $l \wedge m \geq 1$, we have

$$\Xi_n(l, m) = a^l \bar{a}^m \Delta^m \sum_{\gamma=1}^{l \wedge m} \left(-\frac{|b|^2}{|a|^2} \right)^\gamma \binom{l-1}{\gamma-1} \binom{m-1}{\gamma-1} \left(\frac{l-\gamma}{a\gamma} P + \frac{m-\gamma}{\Delta \bar{a}\gamma} Q - \frac{1}{\Delta \bar{b}} R + \frac{1}{b} S \right),$$

by the path counting method [14, 15]. Therefore, for $k \in \{1, \dots, [n/2]\}$, we have

$$\begin{aligned}
P_n^\varphi(n-2k) &= |a|^{2(n-1)} \sum_{\gamma=1}^k \sum_{\delta=1}^k \left(-\frac{|b|^2}{|a|^2} \right)^{\gamma+\delta} \binom{k-1}{\gamma-1} \binom{k-1}{\delta-1} \binom{n-k-1}{\gamma-1} \binom{n-k-1}{\delta-1} \\
&\quad \times \left(\frac{1}{\gamma\delta} \right) \left[\{k^2|a|^2 + (n-k)^2|b|^2 - (\gamma+\delta)(n-k)\} |\alpha|^2 \right. \\
&\quad \quad + \{k^2|b|^2 + (n-k)^2|a|^2 - (\gamma+\delta)k\} |\beta|^2 \\
&\quad \quad + \frac{1}{|b|^2} \left[\{(n-k)\gamma - k\delta + n(2k-n)|b|^2\} a\alpha\bar{b}\bar{\beta} \right. \\
&\quad \quad \quad \left. \left. + \{-k\gamma + (n-k)\delta + n(2k-n)|b|^2\} \bar{a}\bar{\alpha}b\beta + \gamma\delta \right] \right],
\end{aligned}$$

$$\begin{aligned}
P_n^\varphi(-(n-2k)) &= |a|^{2(n-1)} \sum_{\gamma=1}^k \sum_{\delta=1}^k \left(-\frac{|b|^2}{|a|^2} \right)^{\gamma+\delta} \binom{k-1}{\gamma-1} \binom{k-1}{\delta-1} \binom{n-k-1}{\gamma-1} \binom{n-k-1}{\delta-1} \\
&\quad \times \left(\frac{1}{\gamma\delta} \right) \left[\{k^2|b|^2 + (n-k)^2|a|^2 - (\gamma+\delta)k\}|\alpha|^2 \right. \\
&\quad \quad \quad + \{k^2|a|^2 + (n-k)^2|b|^2 - (\gamma+\delta)(n-k)\}|\beta|^2 \\
&\quad \quad \quad + \frac{1}{|b|^2} \left[\{k\gamma - (n-k)\delta - n(2k-n)|b|^2\}a\alpha\bar{b}\bar{\beta} \right. \\
&\quad \quad \quad \left. \left. + \{-(n-k)\gamma + k\delta - n(2k-n)|b|^2\}\bar{a}\bar{\alpha}b\beta + \gamma\delta \right] \right],
\end{aligned}$$

$$\begin{aligned}
P_n^\varphi(n) &= |a|^{2(n-1)} \{ |b|^2|\alpha|^2 + |a|^2|\beta|^2 - (a\alpha\bar{b}\bar{\beta} + \bar{a}\bar{\alpha}b\beta) \}, \\
P_n^\varphi(-n) &= |a|^{2(n-1)} \{ |a|^2|\alpha|^2 + |b|^2|\beta|^2 + (a\alpha\bar{b}\bar{\beta} + \bar{a}\bar{\alpha}b\beta) \}.
\end{aligned}$$

Here $[x]$ denote the integer part of $x \in \mathbb{R}$ where \mathbb{R} is the set of real numbers.

Let $P_n^{\nu,\mu}(x)$ denote the Jacobi polynomial which is orthogonal on $[-1, 1]$ with respect to $(1-x)^\nu(1+x)^\mu$ with $\nu, \mu > -1$. Then the following relation holds:

$$P_n^{\nu,\mu}(x) = \frac{\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(\nu+1)} {}_2F_1(-n, n+\nu+\mu+1; \nu+1; (1-x)/2),$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric series and $\Gamma(z)$ is the gamma function. In general, as for orthogonal polynomials, see [5]. Then we have

$$\sum_{\gamma=1}^k \left(-\frac{|b|^2}{|a|^2} \right)^{\gamma-1} \frac{1}{\gamma} \binom{k-1}{\gamma-1} \binom{n-k-1}{\gamma-1} = \frac{|a|^{-2(k-1)}}{k} P_{k-1}^{1, n-2k}(2|a|^2-1), \quad (3.5)$$

$$\sum_{\gamma=1}^k \left(-\frac{|b|^2}{|a|^2} \right)^{\gamma-1} \binom{k-1}{\gamma-1} \binom{n-k-1}{\gamma-1} = |a|^{-2(k-1)} P_{k-1}^{0, n-2k}(2|a|^2-1). \quad (3.6)$$

By using Eqs. (3.5) and (3.6), we see that for $k \in \{1, \dots, [n/2]\}$,

$$\begin{aligned}
P_n^\varphi(\pm(n-2k)) &= |a|^{2n-4k-2}|b|^4/2 \\
&\quad \times \left[\left\{ \frac{2x^2-2x+1}{x^2} (P^1)^2 - \frac{2}{x} P^1 P^0 + \frac{2}{|b|^2} (P^0)^2 \right\} \right. \\
&\quad \quad \pm \left(\frac{1-2x}{x} \right) \left\{ -\frac{1}{x} \{ (|a|^2 - |b|^2)(|\alpha|^2 - |\beta|^2) + 2(a\alpha\bar{b}\bar{\beta} + \bar{a}\bar{\alpha}b\beta) \} (P^1)^2 \right. \\
&\quad \quad \quad \left. \left. - 2 \left(|\alpha|^2 - |\beta|^2 - \frac{a\alpha\bar{b}\bar{\beta} + \bar{a}\bar{\alpha}b\beta}{|b|^2} \right) P^0 P^1 \right\} \right].
\end{aligned}$$

where $P^i = P_{l-1}^{i, n-2l}(2|a|^2-1)$ ($i = 0, 1$). Let $a(n) \sim b(n)$ means $a(n)/b(n) \rightarrow 1$ as $n \rightarrow \infty$. As it is mentioned in [14, 15], if $n \rightarrow \infty$ with $k/n \in (-(1-|a|)/2, (1+|a|)/2)$, then

$$\begin{aligned}
P^0 &\sim \frac{2|a|^{2k-n}}{\sqrt{\pi n \sqrt{-\Lambda}}} \cos(An+B), \\
P^1 &\sim \frac{2|a|^{2k-n}}{\sqrt{\pi n \sqrt{-\Lambda}}} \sqrt{\frac{x}{(1-x)(1-|a|^2)}} \cos(An+B+\theta),
\end{aligned}$$

where $\Lambda = (1-|a|^2)\{(2x-1)^2-|a|^2\}$, A and B are some constants which are independent of n , and $\theta \in [0, \pi/2]$ is determined by $\cos \theta = \sqrt{(1-|a|^2)/4x(1-x)}$. By these asymptotics and the Riemann-Lebesgue lemma,

we obtain

$$\begin{aligned}
R_{\hat{\alpha}}^{\varphi}(n) &= \frac{1}{1-\hat{\alpha}} \log_2 \left\{ \sum_{x=-n}^n P_n^{\varphi}(x)^{\hat{\alpha}} \right\} \\
&\sim \frac{1}{1-\hat{\alpha}} \log_2 \left\{ \int_{(1-|a|)/2}^{(1+|a|)/2} f(1-2x)^{\hat{\alpha}} dx \times \frac{1}{n^{\hat{\alpha}-1}} \right\} \\
&= \log_2 n + \frac{1}{1-\hat{\alpha}} \log_2 \left\{ \int_{-|a|}^{|a|} f(x)^{\hat{\alpha}} dx \times 2^{\hat{\alpha}-1} \right\} \\
&= \log_2(n/2) + \frac{1}{1-\hat{\alpha}} \log_2 \left\{ \int_{-|a|}^{|a|} f(x)^{\hat{\alpha}} dx \right\}.
\end{aligned}$$

By the same argument, we have

$$T_{\hat{\alpha}}^{\varphi}(n) + \frac{1}{1-\hat{\alpha}} = \frac{1}{1-\hat{\alpha}} \sum_{x=-n}^n P_n^{\varphi}(x)^{\hat{\alpha}} \sim \left(\frac{n}{2}\right)^{1-\hat{\alpha}} \times \frac{1}{1-\hat{\alpha}} \int_{-|a|}^{|a|} f(x)^{\hat{\alpha}} dx,$$

as $n \rightarrow \infty$. This completes the proof.

4 Summary

In this paper, we show a limit theorem for the Tsallis and Rényi entropies of the DTQWs on \mathbb{Z} starting from the origin with arbitrary coin and initial state by using a path counting method. The result shows that asymptotic behaviors of both entropies are strongly dependent on that of scaling limit of the walker's position in any choice of the parameter. It is an interesting future problem that building some novel information theoretic scheme by using DTQWs and these entropies. Also searching for the entropic meaning of the LDP for the DTQW is one of important future problems to be solved.

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